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# Local Euler-Maclaurin expansion of Barvinok valuations and Ehrhart coefficients of a rational polytope

Velleda Baldoni, Nicole Berline and Michèle Vergne

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## 1 Introduction

Let  $\mathbf{p}$  be a rational polytope in  $V = \mathbb{R}^d$  and  $h(x)$  a polynomial function on  $V$ . A classical problem in Integer Programming is to compute the sum of values of  $h(x)$  over the set of integral points of  $\mathbf{p}$ ,

$$S(\mathbf{p}, h) = \sum_{x \in \mathbf{p} \cap \mathbb{Z}^d} h(x).$$

When  $\mathbf{p}$  is dilated by an integer  $n \in \mathbb{N}$ , we obtain a function of  $n$  which is quasi-polynomial, the so-called Ehrhart quasi-polynomial of the pair  $(\mathbf{p}, h)$

$$S(n\mathbf{p}, h) = \sum_{m=0}^{d+N} E_m(\mathbf{p}, h, n) n^m$$

of degree  $d + N$  where  $N = \deg h$ . The coefficients  $E_m(\mathbf{p}, h, n)$  are periodic functions of  $n \in \mathbb{N}$ , with period the smallest integer  $q$  such that  $q\mathbf{p}$  is a lattice polytope.

Replacing  $h(x)$  by an exponential, we are led to study the analytic function on  $V^*$  defined by

$$S(\mathbf{p})(\xi) = \sum_{x \in \mathbf{p} \cap \mathbb{Z}^d} e^{\langle \xi, x \rangle}.$$

If  $\mathbf{p}$  is any rational polyhedron, this sum still makes sense as a meromorphic function defined near 0 and the map  $\mathbf{p} \mapsto S(\mathbf{p})(\xi)$  is a valuation.

In [4], we proved that the meromorphic function  $S(\mathbf{p})(\xi)$  has a local Euler-Maclaurin expansion

$$S(\mathbf{p})(\xi) = \sum_{\mathbf{f}} \mu(\mathbf{t}(\mathbf{p}, \mathbf{f}))(\xi) \int_{\mathbf{f}} e^{\langle \xi, x \rangle} dx.$$

The sum is taken over the set of faces  $\mathbf{f}$  of the polyhedron  $\mathbf{p}$ . For each face  $\mathbf{f}$ , the function  $\mu(\mathbf{t}(\mathbf{p}, \mathbf{f}))(\xi)$  is holomorphic near 0, and it depends only on the transverse cone  $\mathbf{t}(\mathbf{p}, \mathbf{f})$  of  $\mathbf{p}$  along  $\mathbf{f}$ . More precisely, once a rational scalar product is chosen on  $V$ , we define canonically a map  $\mathbf{a} \mapsto \mu(\mathbf{a})$  from the set of rational affine cones  $\mathbf{a}$  in quotient spaces of  $V$ , with values in the space of functions on  $V$  which are analytic near 0, then we prove that these functions satisfy the above formula. The map  $\mathbf{a} \mapsto \mu(\mathbf{a})$  is invariant under lattice translations, equivariant with respect to lattice preserving isometries, and it is a valuation on the set of affine cones with a fixed vertex ([4], Theorems 17 and 18).

It is easy to see that the Ehrhart quasi-polynomial can be computed in terms of the Taylor coefficients of the functions  $\mu(\mathbf{t}(\mathbf{p}, \mathbf{f}))(\xi)$ . For example, if  $\mathbf{p}$  is a lattice polytope and  $h(x) = 1$ , we have ([4], Corollary 28)

$$(1) \quad \text{Card}(n\mathbf{p} \cap \mathbb{Z}^d) = \sum_{\mathbf{f}} \mu(\mathbf{t}(\mathbf{p}, \mathbf{f}))(0) \text{vol}(\mathbf{f}) n^{\dim \mathbf{f}}.$$

Using the valuation property of  $\mu(\mathbf{a})$  and Barvinok's decomposition of a cone into unimodular cones, we thus obtained in [4] an algorithm for computing the Ehrhart quasi-polynomial. It has polynomial length with respect to the input  $(\mathbf{p}, h)$ , when the dimension  $d$  and the degree  $N$  are fixed.

The valuation  $S(\mathbf{p}, h)$  has a natural generalization used by Barvinok in [2], the *mixed* valuation  $S^L(\mathbf{p}, h)$ , where  $L \subseteq V$  is a rational vector subspace. Denote the projected lattice on  $V/L$  by  $\Lambda_L$ . For a polytope  $\mathbf{p} \subset V$  and a polynomial  $h(x)$

$$S^L(\mathbf{p}, h) = \sum_{y \in \Lambda_{V/L}} \int_{\mathbf{p} \cap (y+L)} h(x) dx.$$

In other words, the polytope  $\mathbf{p}$  is sliced along lattice affine subspaces parallel to  $L$  and the integrals of  $h$  over the slices are added up. For  $L = V$ , there is only one term and  $S^V(\mathbf{p}, h)$  is just the integral of  $h(x)$  over  $\mathbf{p}$ , while, for  $L = \{0\}$ , we recover  $S(\mathbf{p}, h)$ , the sum of values of  $h(x)$  over the set of integral points of  $\mathbf{p}$ .

In the case  $h(x) = 1$ , we write  $S(\mathfrak{p})$  and  $S^L(\mathfrak{p})$  in place of  $S(\mathfrak{p}, 1)$  and  $S^L(\mathfrak{p}, 1)$ .

Using these mixed valuations, Barvinok gave an algorithm which computes the  $r + 1$  highest degree Ehrhart coefficients of  $S(n\mathfrak{p}) = \text{Card}(n\mathfrak{p} \cap \mathbb{Z}^d)$ , when  $\mathfrak{p}$  is a simplex in  $\mathbb{R}^d$ . Barvinok's algorithm has polynomial length when  $d$  is an input, provided  $r$  is fixed. The method consists in reducing the problem to summations over lattice points in dimension  $\leq r$ .

Barvinok considers particular linear combinations

$$\sum_{L \in \mathcal{L}} \rho(L) S^L(\mathfrak{p}),$$

where  $\mathcal{L}$  is a finite set of rational vector subspaces of  $V$  which is closed under sum, and the coefficients  $\rho(L)$  are integers which satisfy the following relation between characteristic functions:

$$\chi(\cup_{L \in \mathcal{L}} L^\perp) = \sum_{L \in \mathcal{L}} \rho(L) \chi(L^\perp),$$

where  $L^\perp \subseteq V^*$  is the orthogonal of  $L$ . We call a function  $\mathcal{L} \rightarrow \mathbb{Z}$  with this property a patchwork function on  $\mathcal{L}$ .

When  $\mathfrak{p}$  is dilated by an integer  $n$ ,  $S^L(n\mathfrak{p})$  is again given by a quasi-polynomial in  $n$ , as is a linear combination

$$\sum_{L \in \mathcal{L}} \rho(L) S^L(n\mathfrak{p}) = \sum_{m=0}^d \nu_m(\mathfrak{p}, n) n^m.$$

The main theoretical result of [2], Theorem (1.3), is the following: if  $\mathcal{L}$  is a family of subspaces which is closed under sum and contains the vector subspace  $\text{lin}(\mathfrak{f})$  parallel to  $\mathfrak{f}$ , for every face  $\mathfrak{f}$  of codimension  $\leq r$  of  $\mathfrak{p}$ , and if  $\rho$  is a patchwork function on  $\mathcal{L}$ , then the  $r + 1$  highest degree coefficients  $\nu_m(\mathfrak{p}, n)$ , for  $m = d, \dots, d - r$ , are equal to the corresponding Ehrhart coefficients of  $S(n\mathfrak{p}) = \text{Card}(n\mathfrak{p} \cap \mathbb{Z}^d)$ .

In the present article, we introduce the meromorphic functions which extend  $S^L(\mathfrak{p})$ . For any polyhedron  $\mathfrak{p}$ ,

$$S^L(\mathfrak{p})(\xi) = \sum_{y \in \Lambda_{V/L}} \int_{\mathfrak{p} \cap (y+L)} e^{\langle \xi, x \rangle} dx$$

is defined as a meromorphic function near  $\xi = 0$ . We show that  $S^L(\mathbf{p})(\xi)$  also enjoys a local Euler-Maclaurin expansion (Theorem 8),

$$S^L(\mathbf{p})(\xi) = \sum_{\mathbf{f}} \mu^L(\mathbf{t}(\mathbf{p}, \mathbf{f}))(\xi) \int_{\mathbf{f}} e^{\langle \xi, x \rangle} dx.$$

Furthermore, for a linear combination of Barvinok type, if  $\mathbf{f}$  is a face of  $\mathbf{p}$  such that  $\text{lin}(\mathbf{f}) \in \mathcal{L}$ , we prove that the  $\mathbf{f}$ -term in the Euler-Maclaurin expansions of

$$S^{\mathcal{L}, \rho}(\mathbf{p})(\xi) = \sum_{L \in \mathcal{L}} \rho(L) S^L(\mathbf{p})(\xi)$$

and of the *usual* valuation  $S(\mathbf{p})(\xi)$  are equal (Theorem 17):

$$\sum_{L \in \mathcal{L}} \rho(L) \mu^L(\mathbf{t}(\mathbf{p}, \mathbf{f}))(\xi) = \mu(\mathbf{t}(\mathbf{p}, \mathbf{f}))(\xi).$$

This is the main result of the present article. From the relation between Ehrhart quasi-polynomials and Euler-Maclaurin expansions, it implies Barvinok's Theorem (1.3).

Actually, we derive from Theorem 17 another computation of the  $r + 1$  highest coefficients of the Ehrhart quasi-polynomial for a pair  $(\mathbf{p}, h)$ , based on Brion's decomposition of a polytope into cones, in the line of [1] and [6].

For each vertex  $s$  of  $\mathbf{p}$ , let  $\mathbf{c}_s$  be the cone of feasible directions of  $\mathbf{p}$  at  $s$ . Instead of the full family  $\mathcal{L}$  generated by taking sums of the subspaces  $\text{lin}(\mathbf{f})$ , when  $\mathbf{f}$  runs over the set of faces of codimension  $\leq r$  of the polytope  $\mathbf{p}$ , we consider, for each vertex  $s$  of  $\mathbf{p}$ , the family  $\mathcal{L}_s$  generated by faces of  $\mathbf{c}_s$  of codimension  $\leq r$ . The point in taking a family which depends on  $s$  lies in the case where  $\mathbf{p}$  is simplicial. Then  $\mathcal{L}_s$  consists only of the spaces  $\text{lin}(\mathbf{f})$  where  $\mathbf{f}$  is a face of  $\mathbf{c}_s$  of codimension  $\leq r$ , as this set is already closed under sum. Moreover the coefficients  $\rho(L)$  are just signed binomial coefficients (Lemma 15), and the computation of  $S^L(\mathbf{c}_s)$  is easier when  $L$  is parallel to a face of  $\mathbf{c}_s$  (Example 6).

Let us describe our method in the simpler case of a lattice polytope  $\mathbf{p}$  and polynomial  $h(x) = 1$ . By Brion's theorem, we have

$$S(\mathbf{p})(\xi) = \sum_s e^{\langle \xi, s \rangle} S(\mathbf{c}_s)(\xi).$$

For each vertex  $s$ , let  $\rho_s : \mathcal{L}_s \rightarrow \mathbb{Z}$  be a patchwork function. We define

$$\mathcal{B}(\mathbf{p})(\xi) = \sum_s e^{\langle \xi, s \rangle} S^{\mathcal{L}_s, \rho_s}(\mathbf{c}_s)(\xi).$$

For the dilated polytope  $n\mathbf{p}$ , we have

$$S(n\mathbf{p})(\xi) = \sum_s e^{n\langle \xi, s \rangle} S(\mathbf{c}_s)(\xi) = \sum_{m \geq 0} \frac{n^m}{m!} \sum_s \langle \xi, s \rangle^m S(\mathbf{c}_s)(\xi).$$

Hence, the meromorphic function  $\frac{1}{m!} \sum_s \langle \xi, s \rangle^m S(\mathbf{c}_s)(\xi)$  is actually regular at  $\xi = 0$  and its value at  $\xi = 0$  is the  $m$ th Ehrhart coefficient of  $\mathbf{p}$ .

We have similarly

$$\mathcal{B}(n\mathbf{p})(\xi) = \sum_{m \geq 0} \frac{n^m}{m!} \sum_s \langle \xi, s \rangle^m S^{\mathcal{L}_s, \rho_s}(\mathbf{c}_s)(\xi).$$

The meromorphic functions  $S^L(\mathbf{c}_s)(\xi)$  and  $S^{\mathcal{L}_s, \rho_s}(\mathbf{c}_s)(\xi)$  have a special form: they can be written as the quotient of an analytic function by a product of  $d' \leq d$  linear forms. Such a function  $\phi$  has an expansion into rational functions  $\phi = \sum_{j \geq -d} \phi_{[j]}$  where  $\phi_{[j]}$  is homogeneous of total degree  $j$ .

Now it follows from our main theorem that, for  $m \geq d - r$ , we have

$$S(\mathbf{c}_s)_{[-m]}(\xi) = S^{\mathcal{L}_s, \rho_s}(\mathbf{c}_s)_{[-m]}(\xi),$$

hence the zero degree terms of  $\sum_s \langle \xi, s \rangle^m S(\mathbf{c}_s)(\xi)$  and  $\sum_s \langle \xi, s \rangle^m S^{\mathcal{L}_s, \rho_s}(\mathbf{c}_s)(\xi)$  are equal. Therefore the latter is also analytic at  $\xi = 0$  and its value at  $\xi = 0$  is the  $m$ th Ehrhart coefficient of  $\mathbf{p}$ . This is the content of Theorem 20.

Thus, besides taking care of any polynomial  $h(x)$ , not only  $h(x) = 1$ , this method to compute the  $r + 1$  highest degree Ehrhart coefficients for the pair  $(\mathbf{p}, h)$  leads to a simpler algorithm than the one proposed in [2]. When  $\mathbf{p}$  is a rational simplex, the contributions of the terms of the form  $S^L(\mathbf{c}_s)(\xi)$  when  $L \in \mathcal{L}_s$  are immediately reduced to the computation of a function  $S(\mathbf{a})$  with  $\mathbf{a}$  a simplicial cone of dimension smaller or equal to  $r$ .

There is also another possible implementation of an algorithm to compute the  $r + 1$  Ehrhart highest degree coefficients for the pair  $(\mathbf{p}, h)$  based on the results of [4]. As seen from Equation (1), this involves the computation of the analytic function  $\mu(\mathbf{t}(\mathbf{p}, \mathbf{f}))$ , also associated to simplicial cones in dimension smaller or equal to  $r$ . We plan to compare the implementation of both methods in the near future.

## 2 Local Euler-Maclaurin expansion of a mixed valuation $S^L$

We consider a rational vector space  $V$  of dimension  $d$ , that is to say a finite dimensional real vector space with a lattice denoted by  $\Lambda_V$  or simply  $\Lambda$ . We will need to consider subspaces and quotient spaces of  $V$ , this is why we cannot just let  $V = \mathbb{R}^d$  and  $\Lambda = \mathbb{Z}^d$ . By lattice, we mean a discrete additive subgroup of  $V$  which generates  $V$  as a vector space. Hence, a lattice is generated by a basis of the vector space  $V$ . A basis of  $V$  which is a  $\mathbb{Z}$ -basis of  $\Lambda_V$  is called an integral basis. The elements of  $\Lambda$  are called integral. An element  $x \in V$  is called rational if  $qx \in \Lambda$  for some integer  $q \neq 0$ . The space of rational points in  $V$  is denoted by  $V_{\mathbb{Q}}$ . A subspace  $L$  of  $V$  is called rational if  $L \cap \Lambda$  is a lattice in  $L$ . If  $L$  is a rational subspace, the image of  $\Lambda$  in  $V/L$  is a lattice in  $V/L$ , so that  $V/L$  is a rational vector space. We will call the image of  $\Lambda$  in  $V/L$  the projected lattice.

**Example 1** Let  $V = \mathbb{R}^2$  with standard lattice  $\mathbb{Z}^2$ . Let  $v_1, v_2$  be two primitive integral independent vectors. Using an integral basis with first basis vector  $v_1$ , a straightforward computation shows that the projected lattice on  $\mathbb{R}^2/\mathbb{R}v_1$  is  $\mathbb{Z}\frac{\bar{v}_2}{\det(v_1, v_2)}$ , where  $\bar{v}_2$  is the projection of  $v_2$  on  $\mathbb{R}^2/\mathbb{R}v_1$ .

A rational space  $V$ , with lattice  $\Lambda$ , has a canonical Lebesgue measure, for which  $V/\Lambda$  has measure 1. An affine subspace  $L$  of  $V$  is called rational if it is a translate of a rational subspace by a rational element. It is similarly provided with a canonical Lebesgue measure. We will denote this measure by  $dm_L$ .

We will denote elements of  $V$  by latin letters  $x, y, v, \dots$  and elements of  $V^*$  by greek letters  $\xi, \alpha, \dots$ . We denote the duality bracket by  $\langle \xi, x \rangle$ .

If  $S$  is a subset of  $V$ , we denote by  $\langle S \rangle$  the affine subspace generated by  $S$ . If  $S$  consists of rational points, then  $\langle S \rangle$  is rational. Remark that  $\langle S \rangle$  may contain no integral point. We denote by  $\text{lin}(S)$  the vector subspace of  $V$  parallel to  $\langle S \rangle$ .

If  $S$  is a subset of  $V$ , we denote by  $S^\perp$  the subspace of  $V^*$  orthogonal to  $S$ :

$$S^\perp = \{\xi \in V^* ; \langle \xi, x \rangle = 0 \text{ for all } x \in S\}.$$

If  $L$  is a subspace of  $V$ , the dual space  $(V/L)^*$  is canonically identified with the subspace  $L^\perp \subset V^*$ .

A convex rational polyhedron  $\mathbf{p}$  in  $V$  (we will simply say polyhedron) is, by definition, the intersection of a finite number of half spaces bounded by rational affine hyperplanes. We say that  $\mathbf{p}$  is solid (in  $V$ ) if  $\langle \mathbf{p} \rangle = V$ . A polytope  $\mathbf{p}$  is a **compact** polyhedron.

The set of non negative real numbers is denoted by  $\mathbb{R}_+$ . A convex rational cone  $\mathbf{c}$  in  $V$  is a closed convex cone  $\sum_{i=1}^k \mathbb{R}_+ v_i$  which is generated by a finite number of elements  $v_i$  of  $V_{\mathbb{Q}}$ . In this article, we simply say cone instead of convex rational cone.

An affine (rational) cone  $\mathbf{a}$  is, by definition, the translate of a cone in  $V$  by a rational element  $s \in V_{\mathbb{Q}}$ . This cone is uniquely defined by  $\mathbf{a}$ .

A cone  $\mathbf{c}$  is called simplicial if it is generated by independent elements of  $V_{\mathbb{Q}}$ . A simplicial cone  $\mathbf{c}$  is called unimodular if it is generated by independent integral vectors  $v_1, \dots, v_k$  such that  $\{v_1, \dots, v_k\}$  can be completed in an integral basis of  $V$ . An affine cone  $\mathbf{a}$  is called simplicial (resp. simplicial unimodular) if it is the translate of a simplicial (resp. simplicial unimodular) cone.

An affine cone  $\mathbf{a}$  is called pointed if it does not contain any straight line.

The set of faces of an affine cone  $\mathbf{a}$  is denoted by  $\mathcal{F}(\mathbf{a})$ . If  $\mathbf{a}$  is pointed, then the vertex of  $\mathbf{a}$  is the unique face of dimension 0, while  $\mathbf{a}$  is the unique face of maximal dimension  $\dim \mathbf{a}$ .

Let us recall the definition of the *transverse cone*  $\mathbf{t}(\mathbf{p}, \mathbf{f})$  of a polyhedron  $\mathbf{p}$  along one of its faces  $\mathbf{f}$ . Let  $x$  be a point in the relative interior of  $\mathbf{f}$ . The cone of feasible directions of  $\mathbf{p}$  at  $x$  is the set  $\mathbf{c}(\mathbf{p}, \mathbf{f}) := \{v \in V ; x + \epsilon v \in \mathbf{p} \text{ for } \epsilon > 0 \text{ small enough}\}$ . It does not depend on the choice of  $x$ . We denote the projection  $V \rightarrow V/\text{lin}(\mathbf{f})$  by  $\pi_{\mathbf{f}}$ . Then  $\mathbf{t}(\mathbf{p}, \mathbf{f})$  is the image  $\pi_{\mathbf{f}}(\mathbf{f} + \mathbf{c}(\mathbf{p}, \mathbf{f}))$  of the affine cone  $\mathbf{f} + \mathbf{c}(\mathbf{p}, \mathbf{f})$  in  $V/\text{lin}(\mathbf{f})$ . It is a solid pointed **affine** cone in the quotient space  $V/\text{lin}(\mathbf{f})$  with vertex  $\pi_{\mathbf{f}}(\langle \mathbf{f} \rangle)$ . In particular, if  $v$  is a vertex of  $\mathbf{p}$ , the transverse cone  $\mathbf{t}(\mathbf{p}, v)$  coincides with the supporting cone  $v + \mathbf{c}(\mathbf{p}, v) \subset V$ .

If  $\mathbf{a}$  is an affine cone and  $\mathbf{f}$  is a face of  $\mathbf{a}$ , then  $\mathbf{c}(\mathbf{a}, \mathbf{f}) = \mathbf{a} + \text{lin}(\mathbf{f})$  and the transverse cone  $\mathbf{t}(\mathbf{a}, \mathbf{f})$  of  $\mathbf{a}$  along  $\mathbf{f}$  is just the projection  $\pi_{\mathbf{f}}(\mathbf{a})$  of  $\mathbf{a}$  on  $V/\text{lin}(\mathbf{f})$ .

**Definition 2** Denote by  $\mathcal{H}(V^*)$  the ring of analytic functions around  $0 \in V^*$ . Denote by  $\mathcal{M}(V^*)$  the ring of meromorphic functions defined around  $0 \in V^*$  and by  $\mathcal{M}_{\ell}(V^*) \subset \mathcal{M}(V^*)$  the subring consisting of those meromorphic functions  $\phi(\xi)$  such that there exists a product of linear forms  $D(\xi)$  with

$$D(\xi)\phi(\xi) \in \mathcal{H}(V^*).$$



A function  $\phi(\xi) \in \mathcal{M}_\ell(V^*)$  has a unique expansion into homogeneous rational functions

$$\phi(\xi) = \sum_{m \gg -\infty} \phi_{[m]}(\xi)$$

where  $m$  is the total degree.

If  $P$  is a homogeneous polynomial on  $V^*$  of degree  $p$ , and  $D$  a product of  $r$  linear forms, then  $\frac{P}{D}$  is an element in  $\mathcal{M}_\ell(V^*)$  homogeneous of degree  $m = p - r$ .

Let us recall the definition of the function  $I(\mathbf{p}) \in \mathcal{M}_\ell(V^*)$  associated to a polyhedron  $\mathbf{p}$ , (see for instance the survey [3]).

**Proposition 3** *There exists a map  $I$  which to every polyhedron  $\mathbf{p} \subset V$  associates a meromorphic function with rational coefficients  $I(\mathbf{p}) \in \mathcal{M}_\ell(V^*)$ , so that the following properties hold:*

- (a) *If  $\mathbf{p}$  contains a straight line, then  $I(\mathbf{p}) = 0$ .*
- (b) *If  $\xi \in V^*$  is such that  $|e^{\langle \xi, x \rangle}|$  is integrable over  $\mathbf{p}$ , then*

$$I(\mathbf{p})(\xi) = \int_{\mathbf{p}} e^{\langle \xi, x \rangle} dm_{<\mathbf{p}>}(x).$$

- (c) *For every point  $s \in V_{\mathbb{Q}}$ , we have*

$$I(s + \mathbf{p})(\xi) = e^{\langle \xi, s \rangle} I(\mathbf{p})(\xi).$$

- (d) *The map  $I$  is a simple valuation: if the characteristic functions  $\chi(\mathbf{p}_i)$  of a family of polyhedra  $\mathbf{p}_i$  satisfy a linear relation  $\sum_i r_i \chi(\mathbf{p}_i) = 0$ , then the functions  $I(\mathbf{p}_i)$  satisfy the relation*

$$\sum_{\{i, <\mathbf{p}_i>=V\}} r_i I(\mathbf{p}_i) = 0.$$

In the following proposition, we define the *mixed valuation*  $\mathbf{p} \mapsto S^L(\mathbf{p})$  associated to a rational vector subspace  $L \subseteq V$ . To any polyhedron  $\mathbf{p}$ , we associate a *meromorphic function*  $S^L(\mathbf{p})(\xi) \in \mathcal{M}(V^*)$ . If  $\mathbf{p}$  is compact, this function is actually regular at 0, and its value for  $\xi = 0$  is the valuation  $E_{L^\perp}(\mathbf{p})$  considered by Barvinok [2].

We denote by  $\Lambda_{V/L}$  the projection on  $V/L$  of the lattice  $\Lambda$ .

**Proposition 4** *Let  $L \subseteq V$  be a rational subspace. There exists a map  $S^L$  which to every rational polyhedron  $\mathbf{p} \subset V$  associates a meromorphic function with rational coefficients  $S^L(\mathbf{p}) \in \mathcal{M}(V^*)$  so that the following properties hold:*

- (a) *If  $\mathbf{p}$  contains a line, then  $S^L(\mathbf{p})=0$ .*
- (b)

$$(2) \quad S^L(\mathbf{p})(\xi) = \sum_{y \in \Lambda_{V/L}} \int_{\mathbf{p} \cap (y+L)} e^{\langle \xi, x \rangle} dm_L(x),$$

*for every  $\xi \in V^*$  such that the above sum converges.*

- (c) *For every point  $s \in \Lambda$ , we have*

$$S^L(s + \mathbf{p})(\xi) = e^{\langle \xi, s \rangle} S^L(\mathbf{p})(\xi).$$

(d) *The map  $S^L$  is a valuation: if the characteristic functions  $\chi(\mathbf{p}_i)$  of a family of polyhedra  $\mathbf{p}_i$  satisfy a linear relation  $\sum_i r_i \chi(\mathbf{p}_i) = 0$ , then the functions  $S^L(\mathbf{p}_i)$  satisfy the same relation*

$$\sum_i r_i S^L(\mathbf{p}_i) = 0.$$

For  $L = \{0\}$ , we recover the valuation  $S$  given by

$$S(\mathbf{p})(\xi) = \sum_{x \in \mathbf{p} \cap \Lambda} e^{\langle \xi, x \rangle},$$

provided this sum is convergent.

For  $L = V$ , we have  $S^V(\mathbf{p}) = I(\mathbf{p})$ , if  $\mathbf{p}$  is solid, and  $S^V(\mathbf{p}) = 0$  otherwise.

The proof is entirely analogous to the case  $L = \{0\}$ , see Theorem 3.1 in [3], and we omit it.

**Remark 5** *The function  $S^L(\mathbf{p})$  is actually an element of  $\mathcal{M}_\ell(V^*)$ , but we do not prove it at this point. Let  $\mathbf{a}$  be an affine cone and  $\{v_i\}$  the generators of its edges. It will follow from the Euler-Maclaurin expansion of  $S^L(\mathbf{a})$  (Theorem 8) that  $\prod_i \langle \xi, v_i \rangle S^L(\mathbf{a})(\xi)$  is analytic near zero for any  $L$ . It would be interesting to prove it a priori. By Brion's theorem and the valuation property, it follows in particular that  $S^L(\mathbf{p}) \in \mathcal{M}_\ell(V^*)$ .*

**Example 6** Let  $\mathfrak{a}$  be a simplicial affine cone in the space  $V$ , and assume that  $L = \text{lin}(\mathfrak{f}_1)$  for some face  $\mathfrak{f}_1$  of  $\mathfrak{a}$ . In this case,  $S^L(\mathfrak{a})(\xi)$  decomposes as product of an integral and a discrete sum. For simplicity, assume that  $\mathfrak{a}$  is solid. Let  $\mathfrak{f}_2$  be the face of  $\mathfrak{a}$  such  $V = \text{lin}(\mathfrak{f}_1) \oplus \text{lin}(\mathfrak{f}_2)$ . We write  $x = x_1 + x_2$  and  $\xi = \xi_1 + \xi_2$  for the corresponding decompositions of  $x \in V$  and  $\xi \in V^*$ . Then  $\mathfrak{a} = \mathfrak{a}_1 + \mathfrak{a}_2$  where  $\mathfrak{a}_i$  is a simplicial affine cone in  $\text{lin}(\mathfrak{f}_i)$ . Let us denote by  $\Lambda_2$  the projected lattice in  $V/\text{lin}(\mathfrak{f}_1) \sim \text{lin}(\mathfrak{f}_2)$ . From (2), we obtain immediately

$$(3) \quad S^L(\mathfrak{a})(\xi_1 + \xi_2) = I(\mathfrak{a}_1)(\xi_1) \sum_{x_2 \in \mathfrak{a}_2 \cap \Lambda_2} e^{\langle \xi_2, x_2 \rangle}.$$

Notice that the lattice  $\Lambda_2$  is usually bigger than  $\Lambda \cap \text{lin}(\mathfrak{f}_2)$ .

**Example 7** Let  $V = \mathbb{R}^2$  with the standard lattice. We compute  $S^L(\mathfrak{a})$  when  $\mathfrak{a}$  is a cone and  $L$  is a line. Let  $\mathfrak{a} = \mathbb{R}_+ v_1 + \mathbb{R}_+ v_2$ , where  $v_1, v_2$  are two linearly independent primitive integral vectors.

(a) Assume that  $L$  is the line supporting an edge of  $\mathfrak{a}$ , say  $L = \mathbb{R}v_1$ . We identify  $V/L$  to  $\mathbb{R}v_2$ . The projected lattice is  $\Lambda_2 = \mathbb{Z} \frac{v_2}{\det(v_1, v_2)}$ , (Example 1), hence, by (3) in Example 6, we have

$$(4) \quad S^L(\mathfrak{a})(\xi) = -\frac{1}{\langle \xi, v_1 \rangle} \frac{1}{1 - e^{\frac{\langle \xi, v_2 \rangle}{\det(v_1, v_2)}}}.$$

(b) Assume now that  $L$  is transverse to both edges of  $\mathfrak{a}$ . Assume that  $\det(v_1, v_2) > 0$ . Let  $L = \mathbb{R}u$  where  $u$  is a primitive integral vector chosen so that  $\det(u, v_2) > 0$ . Let  $\mathfrak{a}_i = \mathbb{R}_+ u + \mathbb{R}_+ v_i$  for  $i = 1, 2$ . We decompose the characteristic function of the cone  $\mathfrak{a}$  as  $\chi(\mathfrak{a}) = \chi(\mathfrak{a}_2) + \chi(\mathfrak{a}_1) - \chi(\mathbb{R}_+ u)$  or  $\chi(\mathfrak{a}) = \chi(\mathfrak{a}_2) - \chi(\mathfrak{a}_1) + \chi(\mathbb{R}_+ v_1)$ , depending on whether  $u$  belongs to  $\mathfrak{a}$  or not. Using the valuation property, case (a) and the relation

$$\frac{1}{1 - e^x} + \frac{1}{1 - e^{-x}} = 1,$$

we obtain in both cases

$$(5) \quad S^L(\mathfrak{a})(\xi) = -\frac{1}{\langle \xi, u \rangle} \left( \frac{1}{1 - e^{\frac{\langle \xi, v_2 \rangle}{\det(u, v_2)}}} - \frac{1}{1 - e^{\frac{\langle \xi, v_1 \rangle}{\det(u, v_1)}}} \right).$$

In this example, we see that  $\langle \xi, v_1 \rangle \langle \xi, v_2 \rangle S^L(\mathfrak{a})(\xi)$  is indeed analytic near  $\xi = 0$ .

In the following theorem and its applications, we will consider the functions  $S^L(\mathbf{p})$  when the space  $V$  is replaced with a quotient space  $W$ . We denote by  $\mathcal{C}(W)$  the set of affine cones in  $W$ . Thus if  $\mathbf{a} \in \mathcal{C}(W)$ , and  $L$  a rational subspace of  $W$ , the function  $S^L(\mathbf{a})$  is a meromorphic function on  $W^*$ . We are going to show that the function  $S^L(\mathbf{a})$  has a local Euler-Maclaurin expansion, which generalizes the case  $L = \{0\}$  of [4].

**Theorem 8** *Let  $V$  be a rational space and  $Q$  a rational scalar product on  $V^*$ . There exists a unique family of maps  $\mu_W^L$ , indexed by pairs  $(W, L)$  where  $W$  is a rational quotient space of  $V$  and  $L$  is a rational vector subspace of  $W$  such that the family enjoys the following properties:*

- (a)  $\mu_W^L$  maps  $\mathcal{C}(W)$  to  $\mathcal{H}(W^*)$ , the space of analytic functions on  $W^*$ .
- (b) If  $W = \{0\}$ , then  $\mu_{\{0\}}^{\{0\}}(\{0\}) = 1$ .
- (c) For  $\dim W > 0$  and  $L = W$ , then  $\mu_W^W(\mathbf{a}) = 0$ .
- (d) If the affine cone  $\mathbf{a} \in \mathcal{C}(W)$  contains a straight line, then  $\mu_W^L(\mathbf{a}) = 0$ .
- (e) For any affine cone  $\mathbf{a}$  in  $W$ , the following formula holds

$$S^L(\mathbf{a}) = \sum_{\mathfrak{f} \in \mathcal{F}(\mathbf{a})} \mu_{W/\text{lin}(\mathfrak{f})}^{L+\text{lin}(\mathfrak{f})/\text{lin}(\mathfrak{f})}(\mathbf{t}(\mathbf{a}, \mathfrak{f})) I(\mathfrak{f})$$

where the sum is over all faces of the cone  $\mathbf{a}$ .

In this last formula, the function  $\mu_{W/\text{lin}(\mathfrak{f})}^{L+\text{lin}(\mathfrak{f})/\text{lin}(\mathfrak{f})}(\mathbf{t}(\mathbf{a}, \mathfrak{f}))$  is considered as a function on  $W^*$  itself by means of the orthogonal projection  $W^* \rightarrow (W/\text{lin}(\mathfrak{f}))^* = (\text{lin}(\mathfrak{f}))^\perp$  with respect to the scalar product on  $W^* \subset V^*$ .

**Proof.** The proof is entirely similar to the case  $L = \{0\}$  studied in [4]. Note that  $\mu_W^{\{0\}}$  coincides with the map denoted by  $\mu_W$  in [4]. The only new item is (c). It follows immediately from the relation  $S^W(\mathbf{a}) = I(\mathbf{a})$ .

**Remark 9** *Let  $\mathbf{a}$  be a solid cone in  $W$ , and let  $\mathfrak{f}$  be a face of  $\mathbf{a}$  such that  $\dim \mathfrak{f} < \dim W$ . If  $L$  is transverse to the face  $\mathfrak{f}$ , that is, if  $L + \text{lin}(\mathfrak{f}) = W$ , then  $\mu_{W/\text{lin}(\mathfrak{f})}^{L+\text{lin}(\mathfrak{f})/\text{lin}(\mathfrak{f})}(\mathbf{t}(\mathbf{a}, \mathfrak{f})) = 0$ . This follows from (c).*

From now on we omit the subscript  $W$ , thus we write  $\mu^L$  in place of  $\mu_W^L$ . The next theorem and its proof are also entirely similar to the case  $L = \{0\}$  in [4].

**Theorem 10** *The analytic functions defined in Theorem 8 have the following properties:*

- (a) *For any  $x \in \Lambda$ , one has  $\mu^L(x + \mathfrak{a}) = \mu^L(\mathfrak{a})$ .*
- (b) *The map  $(\mathfrak{a}, L) \mapsto \mu^L(\mathfrak{a})$  is equivariant with respect to lattice-preserving isometries. In other words, let  $g$  be an isometry of  $W$  which preserves the lattice  $\Lambda_W$ . Then  $\mu^{g(L)}(g(\mathfrak{a}))(^t g^{-1} \xi) = \mu^L(\mathfrak{a})(\xi)$ .*
- (c) *If  $W$  is an orthogonal sum  $W = W_1 \oplus W_2$  of rational spaces,  $L_i \subseteq W_i$  and  $\mathfrak{a}_i$  is an affine cone in  $W_i$  for  $i = 1, 2$ , then*

$$\mu^{L_1 \oplus L_2}(\mathfrak{a}_1 + \mathfrak{a}_2) = \mu^{L_1}(\mathfrak{a}_1) \mu^{L_2}(\mathfrak{a}_2).$$

- (d) *For a fixed  $s \in W_{\mathbb{Q}}$ , the map  $\mathfrak{c} \rightarrow \mu^L(s + \mathfrak{c})$  is a valuation on the set of cones in  $W$ .*
- (e) *Let  $\mathfrak{p} \subset W$  be a rational polyhedron, then*

$$(6) \quad S^L(\mathfrak{p}) = \sum_{\mathfrak{f} \in \mathcal{F}(\mathfrak{p})} \mu^{L + \text{lin}(\mathfrak{f}) / \text{lin}(\mathfrak{f})}(\mathfrak{t}(\mathfrak{p}, \mathfrak{f})) I(\mathfrak{f}).$$

**Example 11** *Let us compute the function  $\mu^L$  for the various transverse cones of Example 7. We define a function  $B(u)$  on  $\mathbb{C}$ , holomorphic near 0, by*

$$B(u) = \frac{1}{1 - e^u} + \frac{1}{u}.$$

*We have*

$$I(\mathfrak{a})(\xi) = \frac{|\det(v_1, v_2)|}{\langle \xi, v_1 \rangle \langle \xi, v_2 \rangle}.$$

*Consider case (b) where  $L$  is transverse to both edges  $\mathfrak{f}_i = \mathbb{R}_+ v_i$  of  $\mathfrak{a}$ .*

*Using the equation  $\det(v_1, v_2)u = \det(u, v_2)v_1 - \det(u, v_1)v_2$ , we have*

$$I(\mathfrak{a})(\xi) = \frac{1}{\langle \xi, u \rangle} \left( \frac{\det(u, v_2)}{\langle \xi, v_2 \rangle} - \frac{\det(u, v_1)}{\langle \xi, v_1 \rangle} \right).$$

*Thus we can rewrite (5) as*

$$S^L(\mathfrak{a}) = \mu^L(\mathfrak{a}) + \sum_{i=1,2} \mu^{L + \text{lin}(\mathfrak{f}_i) / \text{lin}(\mathfrak{f}_i)}((\mathfrak{t}(\mathfrak{a}, \mathfrak{f}_i)) I(\mathfrak{f}_i) + I(\mathfrak{a})),$$

*with*

$$(7) \quad \mu^L(\mathfrak{a})(\xi) = \frac{1}{\langle \xi, u \rangle} \left[ B\left(\frac{\langle \xi, v_1 \rangle}{\det(u, v_1)}\right) - B\left(\frac{\langle \xi, v_2 \rangle}{\det(u, v_2)}\right) \right],$$

$$\mu^{L + \text{lin}(\mathfrak{f}_i) / \text{lin}(\mathfrak{f}_i)}((\mathfrak{t}(\mathfrak{a}, \mathfrak{f}_i))) = 0 \quad \text{for } i = 1, 2,$$

Observe that (7) is indeed regular at  $\xi = 0$ .

In case (a) where  $L = \mathbb{R}v_1$ , we have  $L + \text{lin}(\mathfrak{f}_1)/\text{lin}(\mathfrak{f}_1) = \{0\}$ . Let us assume that  $\det(v_1, v_2) > 0$ . Then we have, by [4],

$$\mu^{\{0\}}(\mathfrak{t}(\mathfrak{a}, \mathfrak{f}_1))(\xi) = B \left( \frac{-C_1 \langle \xi, v_1 \rangle + \langle \xi, v_2 \rangle}{\det(v_1, v_2)} \right)$$

with  $C_1 = \frac{Q(v_1, v_2)}{Q(v_1, v_1)}$ .

As  $I(\mathfrak{f}_1)(\xi) = -\frac{1}{\langle \xi, v_1 \rangle}$ , we can rewrite (4) as

$$S^L(\mathfrak{a}) = \mu^L(\mathfrak{a}) + \mu^{\{0\}}(\mathfrak{t}(\mathfrak{a}, \mathfrak{f}_1))I(\mathfrak{f}_1) + \mu^{L+\text{lin}(\mathfrak{f}_2)/\text{lin}(\mathfrak{f}_2)}((\mathfrak{t}(\mathfrak{a}, \mathfrak{f}_2))I(\mathfrak{f}_2) + I(\mathfrak{a}),$$

with

$$\mu^L(\mathfrak{a})(\xi) = \frac{1}{\langle \xi, v_1 \rangle} \left[ B \left( \frac{-C_1 \langle \xi, v_1 \rangle + \langle \xi, v_2 \rangle}{\det(v_1, v_2)} \right) - B \left( \frac{\langle \xi, v_2 \rangle}{\det(v_1, v_2)} \right) \right],$$

which is indeed regular at  $\xi = 0$ , and, again,

$$\mu^{L+\text{lin}(\mathfrak{f}_2)/\text{lin}(\mathfrak{f}_2)}((\mathfrak{t}(\mathfrak{a}, \mathfrak{f}_2))) = 0.$$

### 3 Barvinok valuations

Let  $\mathcal{L}$  be a finite family of rational vector subspaces of  $V$ , and let  $\rho(L), L \in \mathcal{L}$ , be a set of rational coefficients. The linear combination  $\sum_{L \in \mathcal{L}} \rho(L) S^L(\mathfrak{p})$  is again a valuation on the set of polyhedra  $\mathfrak{p} \subset V$ , with values in  $\mathcal{M}_\ell(V^*)$ . By taking linear combinations, we obtain a local Euler-Maclaurin expansion for the function  $\sum_{L \in \mathcal{L}} \rho(L) S^L(\mathfrak{p})(\xi)$ .

**Definition 12** Let  $\mathfrak{p} \subset V$  be a polyhedron.

(a) We denote

$$S^{(\mathcal{L}, \rho)}(\mathfrak{p}) = \sum_{L \in \mathcal{L}} \rho(L) S^L(\mathfrak{p}).$$

(b) We define the  $\mathfrak{f}$ -term in the local Euler-Maclaurin expansion of  $S^{(\mathcal{L}, \rho)}(\mathfrak{p})$  to be

$$\mu^{(\mathcal{L}, \rho)}(\mathfrak{t}(\mathfrak{p}, \mathfrak{f})) = \sum_{L \in \mathcal{L}} \rho(L) \mu^{L+\text{lin}(\mathfrak{f})/\text{lin}(\mathfrak{f})}(\mathfrak{t}(\mathfrak{p}, \mathfrak{f})).$$

Thus we have

$$S^{(\mathcal{L}, \rho)}(\mathfrak{p}) = \sum_{\mathfrak{f} \in \mathcal{F}(\mathfrak{p})} \mu^{(\mathcal{L}, \rho)}(\mathfrak{t}(\mathfrak{p}, \mathfrak{f})) I(\mathfrak{f}).$$

We are going to compute the  $\mathfrak{f}$ -term in the case of the following particular linear combinations introduced by Barvinok [2].

**Definition 13** *The valuation  $S^{(\mathcal{L}, \rho)}$  is called a Barvinok valuation if*

- (a) *the family of subspaces  $\mathcal{L}$  is stable under sum,*
- (b)  *$\rho$  is an integer valued function on the set  $\mathcal{L}$  such that the characteristic function of the union of the subspaces  $L^\perp \subseteq V^*$  can be written as the linear combination*

$$(8) \quad \chi(\cup_{L \in \mathcal{L}} L^\perp) = \sum_{L \in \mathcal{L}} \rho(L) \chi(L^\perp).$$

**Definition 14** *We call a function  $\mathcal{L} \rightarrow \mathbb{Z}$  which satisfies (8) a patchwork function on  $\mathcal{L}$ .*

As the set of orthogonal subspaces  $L^\perp \subseteq V^*$  is stable under intersection, a particular function  $\rho_{\mathcal{L}}$  with this property can be computed in terms of the Moebius function of the partially ordered set  $\mathcal{L}$  ([7], vol I, section 3.7), as explained in [2].

Let us compute a patchwork function  $\rho$  in the following case.  $V = \mathbb{R}^d$  with standard basis  $e_i, i = 1, \dots, d$ , and  $\mathcal{L}_{d,q}$  is the set of subspaces  $L_I = \oplus_{i \in I} \mathbb{R}e_i$  with cardinal  $|I| \geq q$ . The function  $\rho_{d,q}$  defined below is actually the one associated to the Moebius function, but we will not need this fact.

We denote the binomial coefficient  $\frac{m!}{k!(m-k)!}$  by  $C_m^k$ .

**Lemma 15** *The function  $\rho_{d,q}$  on  $\mathcal{L}_{d,q}$  defined by*

$$\rho_{d,q}(L_I) = (-1)^{n-q} C_{n-1}^{q-1} \quad \text{if } |I| = n,$$

*satisfies Equation (8).*

**Proof.** If  $e^i$  is the dual basis, the orthogonal space  $L_I^\perp$  is equal to  $\sum_{i \notin I} \mathbb{R}e^i$ . Let  $\xi = \sum_{i=1}^d \xi_i e^i \in \cup_{L \in \mathcal{L}_q} L^\perp$ . Let  $I_0$  be the set of indices  $i \in [1, \dots, d]$  such that  $\xi_i = 0$ . Then  $|I_0| \geq q$ , and  $\xi \in L_I^\perp$  if and only if  $I \subseteq I_0$ . Let  $|I_0| = N$ . The value at  $\xi$  of the right-hand side of (8) is equal to

$$E(N, q) = \sum_{I \subseteq I_0} \rho_{d,q}(L_I) = \sum_{n=q}^N (-1)^{n-q} C_N^n C_{n-1}^{q-1}.$$

We want to prove that  $E(N, q) = 1$ . Writing  $n = q + i$ , we have

$$E(N, q) = \sum_{i=0}^{N-q} (-1)^i \frac{N!}{(q+i)(N-q-i)!i!(q-1)!}.$$

Let us compute  $(q-1)!(E(N+1, q) - E(N, q))$ . This is equal to

$$\begin{aligned} & (-1)^{N+1-q} \frac{(N+1)!}{(N+1)(N+1-q)!} + \sum_{i=0}^{N-q} (-1)^i \frac{1}{(q+i)i!} \left( \frac{(N+1)!}{(N+1-q-i)!} - \frac{N!}{(N-q-i)!} \right) \\ &= N! \sum_{i=0}^{N+1-q} (-1)^i \frac{N!}{i!(N+1-q-i)!} = 0. \end{aligned}$$

We obtain  $E(N, q) = E(q, q) = 1$ .  $\square$

In the case of a Barvinok valuation, it turns out that the  $\mathfrak{f}$ -term in the Euler-Maclaurin expansion of  $S^{(\mathcal{L}, \rho)}(\mathfrak{p})$  coincides with that of  $S(\mathfrak{p})$ , if the vector subspace  $\text{lin}(\mathfrak{f})$  belongs to the set  $\mathcal{L}$ . This is the crucial result of the present article. It is an easy consequence of the following combinatorial lemma.

**Lemma 16** *Let  $\mathcal{L}$  be a finite family of vector subspaces of  $V$ , stable under sum and let  $\rho$  be a patchwork function on  $\mathcal{L}$ .*

(a) *Let  $L_0 \in \mathcal{L}$ . Then*

$$\sum_{\{L \in \mathcal{L}, L \subseteq L_0\}} \rho(L) = 1.$$

(b) *Let  $L_0 \subsetneq L_1$  be two subspaces in the family  $\mathcal{L}$ . Then*

$$\sum_{\{L \in \mathcal{L}, L+L_0=L_1\}} \rho(L) = 0.$$

**Proof.** There exists a  $\xi_0 \in L_0^\perp$  such that, for  $L \in \mathcal{L}$ ,  $\xi_0 \in L^\perp$  if and only if  $L \subseteq L_0$ . We obtain (a) by evaluating both sides of (8) at this particular element  $\xi_0$ . Next, we deduce (b) from (a), by induction on  $\dim L_1 - \dim L_0$ .

For two subspaces  $M \subseteq M'$  in the family  $\mathcal{L}$ , let us denote

$$f(M, M') = \sum_{\{L \in \mathcal{L}, L+M=M'\}} \rho(L).$$



If  $M = M'$ , we have  $f(M, M) = \sum_{\{L \in \mathcal{L}, L \subseteq M\}} \rho(L) = 1$  by (a).

We apply this with  $M = L_1$ . Thus

$$\sum_{\{L \in \mathcal{L}, L \subseteq L_1\}} \rho(L) = 1.$$

In this sum, we group the  $L \in \mathcal{L}$  such that  $L + L_0$  is equal to a given  $M_1 \in \mathcal{L}$  together.

First we consider the case when  $\dim L_1 - \dim L_0 = 1$ . Then  $M_1$  is either  $L_0$  or  $L_1$ , hence we obtain

$$f(L_0, L_0) + f(L_0, L_1) = 1.$$

Since  $f(L_0, L_0) = 1$  by (a), we obtain  $f(L_0, L_1) = 0$  as required.

Next we consider the case when  $\dim L_1 - \dim L_0 > 1$ . We obtain

$$\sum_{\{M_1 \in \mathcal{L}, L_0 \subseteq M_1 \subseteq L_1\}} f(L_0, M_1) = 1.$$

For  $M_1 = L_0$ , we have  $f(L_0, L_0) = 1$  by (a). For  $M_1 \subsetneq L_1$ , we have  $f(L_0, M_1) = 0$  by induction hypothesis. Hence there remains only the term  $f(L_0, L_1)$  which must be equal to 0.  $\square$

We study now the Euler-Maclaurin expansion of a Barvinok valuation.

**Theorem 17** *Let  $\mathfrak{p} \subset V$  be a rational polyhedron and let  $\mathfrak{f}$  be a face of  $\mathfrak{p}$ . Let  $\mathcal{L}$  be a finite family of rational vector subspaces of  $V$ , stable under sum. Let  $\rho$  be a patchwork function on  $\mathcal{L}$ , and let  $S^{(\mathcal{L}, \rho)} = \sum_{L \in \mathcal{L}} \rho(L) S^L$ .*

*Assume that  $\text{lin}(\mathfrak{f})$  belongs to  $\mathcal{L}$ . Then*

$$\mu^{(\mathcal{L}, \rho)}(\mathfrak{t}(\mathfrak{p}, \mathfrak{f})) = \mu^{\{0\}}(\mathfrak{t}(\mathfrak{p}, \mathfrak{f})).$$

*In other words, the  $\mathfrak{f}$ -term in the local Euler-Maclaurin expansion of  $S^{(\mathcal{L}, \rho)}(\mathfrak{p})$  coincides with that of  $S(\mathfrak{p})$ .*

**Proof.** In the sum  $\sum_{L \in \mathcal{L}} \rho(L) \mu^{L + \text{lin}(\mathfrak{f}) / \text{lin}(\mathfrak{f})}(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))$ , we group the terms for which  $L + \text{lin}(\mathfrak{f})$  is equal to a given  $L_1$  together. By Lemma 16 we obtain  $\mu^{\{0\}}(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))$  for  $L_1 = \text{lin}(\mathfrak{f})$  and 0 otherwise.  $\square$

**Corollary 18** *Let  $\mathfrak{p} \subset V$  be a rational polyhedron. Let  $0 \leq k \leq d$ . Let  $\mathcal{L}$  be a finite family of rational vector subspaces of  $V$ , stable under sum, such that  $\text{lin}(\mathfrak{f}) \in \mathcal{L}$  for every  $k$ -dimensional face  $\mathfrak{f}$  of  $\mathfrak{p}$  and let  $\rho$  be a patchwork function on  $\mathcal{L}$ .*

- *Let  $0 < k \leq d$ . Then the meromorphic function*

$$S(\mathfrak{p})(\xi) - S^{(\mathcal{L}, \rho)}(\mathfrak{p})(\xi)$$

*has lowest degree  $\geq -k + 1$ .*

- *Let  $k = 0$ . Then*

$$S(\mathfrak{p})(\xi) = S^{(\mathcal{L}, \rho)}(\mathfrak{p})(\xi).$$

**Proof.** By Theorem 17, the local Euler-Maclaurin expansion of the difference involves only faces of dimension  $< k$ .

$$S(\mathfrak{p})(\xi) - S^{(\mathcal{L}, \rho)}(\mathfrak{p})(\xi) = \sum_{\{\mathfrak{f} \in \mathcal{F}(\mathfrak{p}), \dim \mathfrak{f} < k\}} (\mu^{\{0\}}(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))(\xi) - \mu^{(\mathcal{L}, \rho)}(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))(\xi)) I(\mathfrak{f})(\xi).$$

For a face of dimension  $j$ , the function  $I(\mathfrak{f})(\xi)$  is homogeneous of degree  $-j$ . Multiplied by the *holomorphic* function  $\mu^{\{0\}}(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))(\xi) - \mu^{(\mathcal{L}, \rho)}(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))(\xi)$ , its lowest degree can only increase.  $\square$

Remark that the statement of Corollary 18 above does not involve any scalar product. In the next section, we will show that this corollary implies our main Theorem (Theorem 20).

## 4 Ehrhart quasi-polynomial

Let  $\mathfrak{p}$  be a rational polytope and let  $h(x)$  be a polynomial function on  $V$ . Let

$$S(\mathfrak{p}, h) = \sum_{x \in \mathfrak{p} \cap \mathbb{Z}^d} h(x).$$

When  $\mathfrak{p}$  is dilated by a non negative integer  $n$ , we obtain the Ehrhart quasi-polynomial of the pair  $(\mathfrak{p}, h)$

$$S(n\mathfrak{p}, h) = \sum_{m=0}^{d+N} E_m(\mathfrak{p}, h, n) n^m,$$

where  $N = \deg h$ . The coefficients  $E_m(\mathbf{p}, h, n)$  are periodic functions of  $n \in \mathbb{N}$ , with period the smallest integer  $q$  such that  $q\mathbf{p}$  is a lattice polytope.

If an integer  $r \leq d$  is fixed, and  $h = 1$ , Barvinok [2] proved that the  $r + 1$  highest Ehrhart coefficients  $E_d(\mathbf{p}, 1, n), \dots, E_{d-r}(\mathbf{p}, 1, n)$  of  $S(n\mathbf{p}, 1)$  can be computed in polynomial time with respect to  $d$ , when  $\mathbf{p}$  is a rational simplex.

Let  $L \subseteq V$  be a rational vector subspace. Denote the projected lattice on  $V/L$  by  $\Lambda_L$ . Consider the mixed valuation

$$S^L(\mathbf{p}, h) = \sum_{y \in \Lambda_{V/L}} \int_{\mathbf{p} \cap (y+L)} h(x) dx.$$

As shown by Barvinok, and as we will see here, we can use linear combination of these mixed valuations to approximate  $S(n\mathbf{p}, h)$  when  $n$  is big.

For any polyhedron  $\mathbf{a}$ , we define the meromorphic function  $S^L(\mathbf{a}, h)(\xi) \in \mathcal{M}_\ell(V^*)$  similarly to  $S^L(\mathbf{a}, h)$ . For  $\xi \in V^*$  such that the sum converges, we have

$$S^L(\mathbf{a}, h)(\xi) = \sum_{y \in \Lambda_{V/L}} \int_{\mathbf{a} \cap (y+L)} h(x) e^{\langle \xi, x \rangle} dm_L(x).$$

**Remark 19** *It is clear that  $S^L(\mathbf{a}, h)(\xi) = h(\partial_\xi) \cdot S^L(\mathbf{a})(\xi)$ .*

*If  $\mathbf{p}$  is a polytope, then  $S^L(\mathbf{p}, h)(\xi)$  is regular at  $\xi = 0$  and  $S^L(\mathbf{p}, h)(0) = S^L(\mathbf{p}, h)$ .*

For a family  $\mathcal{L}$  and a function  $L \mapsto \rho(L)$ ,  $L \in \mathcal{L}$ , we define

$$S^{(\mathcal{L}, \rho)}(\mathbf{a}, h)(\xi) = \sum_{L \in \mathcal{L}} \rho(L) S^L(\mathbf{a}, h)(\xi).$$

If  $\mathbf{p}$  is a polytope, and we dilate by  $n \in \mathbb{N}$ , we have again a quasi-polynomial

$$S^{(\mathcal{L}, \rho)}(n\mathbf{p}, h)(0) = \sum_{m=0}^{d+N} E_m(\mathcal{L}, \rho, \mathbf{p}, h, n) n^m.$$

We can replace the quasi-polynomial  $S^{(\mathcal{L}, \rho)}(n\mathbf{p}, h)(0)$  by  $q$  legal polynomials in the variable  $u$ , by splitting  $\mathbb{N}$  into classes modulo  $q$ . Writing  $n = qu + k$ , for  $k = 0, \dots, q - 1$ , we obtain the polynomial function of  $u$ :

$$S^{(\mathcal{L}, \rho)}((qu + k)\mathbf{p}, h)(0) = \sum_{m=0}^{d+N} E_m^{(k)}(\mathcal{L}, \rho, \mathbf{p}, h) u^m.$$

We briefly recall how the usual Ehrhart quasi-polynomial of a polytope can be computed using Brion's theorem. We will then use a similar method in order to compute efficiently the  $r + 1$  highest coefficients only, using Barvinok valuations.

Let  $\mathcal{V}(\mathbf{p})$  be the set of vertices of  $\mathbf{p}$ . For each vertex  $s$ , let  $\mathbf{c}_s$  be the cone of feasible directions of  $\mathbf{p}$  at  $s$ , so that the supporting cone at  $s$  is  $s + \mathbf{c}_s$ . By Brion's theorem [5], we have

$$S(\mathbf{p}, h)(\xi) = \sum_{s \in \mathcal{V}(\mathbf{p})} S(s + \mathbf{c}_s, h)(\xi).$$

Let  $n \in \mathbb{N}$  and consider the dilated polytope  $n\mathbf{p}$ . The supporting cone at the vertex  $ns$  is  $ns + \mathbf{c}_s$ . Let  $q \in \mathbb{N}$  such that  $q\mathbf{p}$  is a lattice polytope and fix  $k \in \mathbb{N}$ ,  $0 \leq k \leq q - 1$ . Let  $n = qu + k$ . As  $qu$  is an integral point, we have

$$S((qu + k)s + \mathbf{c}_s, h)(\xi) = e^{qu\langle \xi, s \rangle} S^{\mathcal{L}_s, \rho_s}(ks + \mathbf{c}_s, h)(\xi).$$

Expanding in powers of  $u$ , we obtain

$$S((qu + k)\mathbf{p}, h)(\xi) = \sum_{m \geq 0} u^m \frac{q^m}{m!} \sum_{s \in \mathcal{V}(\mathbf{p})} \langle \xi, s \rangle^m S(ks + \mathbf{c}_s, h)(\xi).$$

It follows that for each  $m$ , the sum of meromorphic functions

$$\frac{q^m}{m!} \sum_{s \in \mathcal{V}(\mathbf{p})} \langle \xi, s \rangle^m S(ks + \mathbf{c}_s, h)(\xi)$$

is actually analytic. Its value at  $\xi = 0$  is obtained by taking the zero degree term. We obtain

$$S((qu + k)\mathbf{p}, h)(0) = \sum_{m \geq 0} E_m^{(k)}(\mathbf{p}, h) u^m,$$

with

$$E_m^{(k)}(\mathbf{p}, h) = \frac{q^m}{m!} \sum_{s \in \mathcal{V}(\mathbf{p})} \langle \xi, s \rangle^m S(ks + \mathbf{c}_s, h)_{[-m]}(\xi).$$

The right-hand side of this relation, a priori a meromorphic function of  $\xi$ , is actually constant. Moreover, we have  $S(ks + \mathbf{c}_s, h)_{[-m]}(\xi) = 0$  if  $m > d + N$ , hence the Ehrhart quasi-polynomial has degree  $\leq d + N$ .

We apply now Brion's theorem to  $S^{\mathcal{L},\rho}(\mathbf{p}, h)(\xi)$ . We obtain

$$S^{\mathcal{L},\rho}(\mathbf{p}, h)(\xi) = \sum_{s \in \mathcal{V}(\mathbf{p})} S^{\mathcal{L},\rho}(s + \mathbf{c}_s, h)(\xi).$$

For reasons to be explained later on, instead of one family  $\mathcal{L}$ , we take a family of subspaces  $\mathcal{L}_s$  for each vertex  $s$ . Let  $\rho_s : \mathcal{L}_s \rightarrow \mathbb{Z}$  be a function on  $\mathcal{L}_s$ . We denote now by  $(\mathcal{L}, \rho)$  the map  $s \mapsto (\mathcal{L}_s, \rho_s)$ .

We define:

$$(9) \quad \mathcal{B}^{\mathcal{L},\rho}(\mathbf{p}, h)(\xi) = \sum_{s \in \mathcal{V}(\mathbf{p})} S^{\mathcal{L}_s, \rho_s}(s + \mathbf{c}_s, h)(\xi).$$

If the family does not depend on  $s$ ,  $(\mathcal{L}_s, \rho_s) = (\mathcal{L}_0, \rho_0)$  for every vertex  $s$ , then, by Brion's theorem, we have

$$\mathcal{B}^{\mathcal{L},\rho}(\mathbf{p}, h)(\xi) = S^{\mathcal{L}_0, \rho_0}(\mathbf{p}, h)(\xi).$$

We dilate (9). Let  $n = qu + k$ . We obtain

$$\mathcal{B}^{\mathcal{L},\rho}((qu + k)\mathbf{p}, h)(\xi) = \sum_{s \in \mathcal{V}(\mathbf{p})} e^{qu\langle \xi, s \rangle} S^{\mathcal{L}_s, \rho_s}(ks + \mathbf{c}_s, h)(\xi).$$

Expanding in powers of  $u$ , we obtain

$$(10) \quad \mathcal{B}^{\mathcal{L},\rho}((qu + k)\mathbf{p}, h)(\xi) = \sum_{m \geq 0} u^m E_m^{(k)}(\mathcal{L}, \rho, \mathbf{p}, h)(\xi)$$

with

$$(11) \quad E_m^{(k)}(\mathcal{L}, \rho, \mathbf{p}, h)(\xi) = \frac{q^m}{m!} \sum_{s \in \mathcal{V}(\mathbf{p})} \langle \xi, s \rangle^m S^{\mathcal{L}_s, \rho_s}(ks + \mathbf{c}_s, h)(\xi).$$

If the family does not depend on  $s$ ,  $(\mathcal{L}_s, \rho_s) = (\mathcal{L}_0, \rho_0)$  for all vertices, then  $\mathcal{B}^{\mathcal{L},\rho}(\mathbf{p}, h)(\xi) = S^{\mathcal{L}_0, \rho_0}(\mathbf{p}, h)(\xi)$  is analytic near  $\xi = 0$ , and so are the coefficients (11).

On the contrary, if we take a different family  $\mathcal{L}_s$  for each vertex  $s$ , the coefficient  $E_m^{(k)}(\mathcal{L}, \rho, \mathbf{p}, h)(\xi)$  of  $u^m$  in (10) is no longer analytic near  $\xi = 0$ , in general. However, the meromorphic function  $\xi \mapsto E_m^{(k)}(\mathcal{L}, \rho, \mathbf{p}, h)(\xi)$  belongs to  $\mathcal{M}_\ell(V^*)$ , thus it has a term of degree 0 with respect to  $\xi$ , given by

$$(12) \quad E_m^{(k)}(\mathcal{L}, \rho, \mathbf{p}, h)_{[0]}(\xi) = \frac{q^m}{m!} \sum_{s \in \mathcal{V}(\mathbf{p})} \langle \xi, s \rangle^m S^{\mathcal{L}_s, \rho_s}(ks + \mathbf{c}_s, h)_{[-m]}(\xi).$$

For a family  $(\mathcal{L}, \rho)$  as described in the next theorem, it turns out that, for large  $m$ , this zero-degree part  $E_m^{(k)}(\mathcal{L}, \rho, \mathbf{p}, h)_{[0]}(\xi)$  is actually analytic, hence constant, and its value is equal to the  $m$ -th Ehrhart coefficient  $E_m^{(k)}(\mathbf{p}, h)$  of  $S((k + qu)\mathbf{p}, h)(0)$ .

**Theorem 20** *Let  $\mathbf{p}$  be a rational polytope in a rational vector space of dimension  $d$ . For each vertex  $s$  of the polytope  $\mathbf{p}$ , let  $\mathbf{c}_s$  be the cone of feasible directions of  $\mathbf{p}$  at  $s$ , so that the supporting cone at  $s$  is  $s + \mathbf{c}_s$ . For each vertex  $s$ , let  $\mathcal{L}_s$  be a finite family of rational vector subspaces of  $V$ , stable under sum, such that  $\text{lin}(\mathbf{f})$  belongs to  $\mathcal{L}_s$  for every face  $\mathbf{f}$  of codimension  $r$  of the cone  $\mathbf{c}_s$ , and let  $\rho_s$  be a patchwork function on  $\mathcal{L}_s$ . Let  $q \in \mathbb{N}$  such that  $q\mathbf{p}$  is a lattice polytope and fix  $k \in \mathbb{N}$ ,  $0 \leq k \leq q - 1$ . Let  $h(x)$  be a homogeneous polynomial of total degree  $N$ .*

*Then, for  $m \geq d + N - r$ , the zero-degree term  $E_m^{(k)}(\mathcal{L}, \rho, \mathbf{p}, h)_{[0]}(\xi)$  defined by (12) is regular near  $\xi = 0$ , hence constant. Its value is the coefficient  $E_m^{(k)}(\mathbf{p}, h)$  of  $u^m$  in the Ehrhart quasi-polynomial*

$$S((k + qu)\mathbf{p}, h)(0) = \sum_{x \in ((k + qu)\mathbf{p}) \cap \Lambda} h(x) = \sum_{m=0}^{d+N} u^m E_m^{(k)}(\mathbf{p}, h).$$

**Proof.** We first consider the case  $h(x) = 1$ . We have, for every  $m \geq 0$ ,

$$E_m^{(k)}(\mathbf{p}, 1) = \frac{q^m}{m!} \sum_{s \in \mathcal{V}(\mathbf{p})} \langle \xi, s \rangle^m S(ks + \mathbf{c}_s)_{[-m]}(\xi)$$

where the right-hand side is actually a constant function of  $\xi$ . For  $m > d - r - 1$ , we have, by Corollary 18,

$$S(ks + \mathbf{c}_s)(\xi)_{[-m]} = S^{\mathcal{L}_s, \rho_s}(ks + \mathbf{c}_s)_{[-m]}(\xi).$$

This proves the theorem when  $h(x) = 1$ . The case of a non constant polynomial  $h(x)$  is quite similar. If  $h(x) = x_1^{N_1} \dots x_d^{N_d}$ , we just have to replace the meromorphic functions  $S(ks + \mathbf{c}_s)(\xi)$  and  $S^{\mathcal{L}_s, \rho_s}(ks + \mathbf{c}_s)(\xi)$  by their derivatives under  $\partial_{\xi_1}^{N_1} \dots \partial_{\xi_d}^{N_d}$ .  $\square$

If for each vertex  $s$ , we take  $\mathcal{L}_s = \mathcal{L}$ , the full collection generated by all  $r$  codimensional faces of  $\mathbf{p}$ , we obtain Corollary 21 below, that is Barvinok's theorem [2], with an extension to the sum of values of any polynomial  $h(x)$  over the set of integral points of a rational polytope (Barvinok considers only the case  $h(x) = 1$ ).

**Corollary 21** *Let  $\mathbf{p} \subset V$  be a rational polytope and let  $h(x)$  be a polynomial function on  $V$ . Let  $\mathcal{L}$  be a finite family of rational vector subspaces of  $V$ , stable under sum. Assume that  $\text{lin}(\mathbf{f})$  belongs to  $\mathcal{L}$  for every face  $\mathbf{f}$  of codimension  $r$  of  $\mathbf{p}$ . Let  $\rho$  be a patchwork function on  $\mathcal{L}$  and let  $S^{\mathcal{L},\rho} = \sum_{L \in \mathcal{L}} \rho(L) S^L$ . Then the  $r + 1$  highest Ehrhart coefficients of  $S(t\mathbf{p}, h)(0)$  and  $S^{\mathcal{L},\rho}(t\mathbf{p}, h)(0)$  are equal.*

The point in taking a family  $\mathcal{L}_s$  which depends on the vertex  $s$  lies in the case where  $\mathbf{p}$  is *simplicial*. In this case, we can take  $\mathcal{L}_s$  to be just the set of subspaces  $\text{lin}(\mathbf{f})$ , for all faces  $\mathbf{f}$  of codimension  $\leq r$  of the supporting cone  $\mathbf{c}_s$  at vertex  $s$ . This family is stable under sum. Moreover the patchwork function on  $\mathcal{L}_s$  is simple, (Lemma 15) and the computation of the function  $S^L(k\mathbf{s} + \mathbf{c}_s)(\xi)$ , when  $L \in \mathcal{L}_s$ , is immediately reduced (Example 6) to the computation of a function  $S(\mathbf{a})(\xi)$  for a simplicial cone  $\mathbf{a}$  in a rational vector space of dimension **smaller or equal** than  $r$ . When  $\mathbf{p}$  is a simplex, we obtain in this way a method for computing the  $r + 1$  highest Ehrhart coefficients for the pair  $(\mathbf{p}, h)$ .

## 5 Local Euler-Maclaurin formula for mixed sums

Finally in this last section, we discuss an application of the existence of the coefficients  $\mu^L$  (Theorem 8) in the line of [4].

Let  $\mathbf{p}$  be a rational polytope in a rational vector space  $V$  of dimension  $d$  and let  $h(x)$  be a polynomial function on  $V$ . Let  $L$  be a rational subspace of  $V$ . Consider the mixed sum

$$S^L(\mathbf{p}, h) = \sum_{y \in \Lambda_{V/L}} \int_{\mathbf{p} \cap (y+L)} h(x) dm_L(x).$$

As in [4], we associate to the analytic function  $\mu^L(\mathbf{t}(\mathbf{p}, \mathbf{f}))$  a constant coefficients differential operator (of infinite order) on  $V$ .

**Definition 22** *Let  $\mathbf{f}$  be a face of  $\mathbf{p}$ . We denote by  $D^L(\mathbf{p}, \mathbf{f})$  the differential operator on  $V$  associated to analytic function  $\mu^L(\mathbf{t}(\mathbf{p}, \mathbf{f}))$ :*

$$D^L(\mathbf{p}, \mathbf{f})(\partial_\xi) \cdot e^{\langle \xi, x \rangle} = \mu^L(\mathbf{t}(\mathbf{p}, \mathbf{f}))(\xi) e^{\langle \xi, x \rangle}.$$

The operators  $D^L(\mathbf{p}, \mathfrak{f})$  are **local**, that is they depend only of the transverse cone  $\mathbf{t}(\mathbf{p}, \mathfrak{f})$  of  $\mathbf{p}$  along  $\mathfrak{f}$ , and they involve only derivatives in directions orthogonal to the face  $\mathfrak{f}$ . We can state the following theorem with the same proof as in [4].

**Theorem 23** (*Local Euler-Maclaurin formula*) *Let  $\mathbf{p}$  be a polytope in  $V$ . For any polynomial function  $h(x)$  on  $V$ , we have*

$$(13) \quad S^L(\mathbf{p}, h) = \sum_{\mathfrak{f} \in \mathcal{F}(\mathbf{p})} \int_{\mathfrak{f}} D^L(\mathbf{p}, \mathfrak{f}) \cdot h$$

where the integral on the face  $\mathfrak{f}$  is taken with respect to the Lebesgue measure on  $\langle \mathfrak{f} \rangle$  defined by the lattice  $\Lambda \cap \text{lin}(\mathfrak{f})$ .

In particular, for  $h = 1$ , we obtain

$$(14) \quad S^L(\mathbf{p}, 1) = \sum_{\mathfrak{f} \in \mathcal{F}(\mathbf{p})} \mu^L(\mathbf{t}(\mathbf{p}, \mathfrak{f}))(0) \text{vol}(\mathfrak{f}).$$

Let us dilate the polytope  $\mathbf{p}$  by a non negative integer  $n$ . If  $\mathfrak{f}$  is a face of  $\mathbf{p}$ , let  $q_{\mathfrak{f}}$  be the smallest positive integer such that  $q_{\mathfrak{f}} \langle \mathfrak{f} \rangle$  contains integral points. Define  $D(\mathbf{p}, \mathfrak{f}, n) = D(n\mathbf{p}, n\mathfrak{f})$ , if  $n > 0$ , and  $D(\mathbf{p}, \mathfrak{f}, 0) = D(q_{\mathfrak{f}}\mathbf{p}, q_{\mathfrak{f}}\mathfrak{f})$ . The function  $n \mapsto D(\mathbf{p}, \mathfrak{f}, n)$  is periodic of period  $q_{\mathfrak{f}}$ .

**Proposition 24** *Let  $\mathbf{p}$  be a rational polytope and  $h$  a polynomial function of degree  $N$  on  $V$ . Then, for any integer  $n \geq 0$ , we have*

$$(15) \quad S^L(n\mathbf{p}, h) = \sum_{\mathfrak{f} \in \mathcal{F}(\mathbf{p})} \int_{n\mathfrak{f}} D^L(\mathbf{p}, \mathfrak{f}, n) \cdot h.$$

Furthermore, if  $\mathfrak{f} \in \mathcal{F}(\mathbf{p})$ , we have

$$\int_{n\mathfrak{f}} D^L(\mathbf{p}, \mathfrak{f}, n) \cdot h = \sum_{i=\dim \mathfrak{f}}^{\dim \mathfrak{f} + N} E_i(\mathbf{p}, h, \mathfrak{f}, n) n^i$$

where the coefficients  $E_i(\mathbf{p}, h, \mathfrak{f}, n)$  are periodic with period  $q_{\mathfrak{f}}$ .

Hence the Ehrhart coefficients are given by

$$E_m^L(\mathbf{p}, h, n) = \sum_{\mathfrak{f}, \dim \mathfrak{f} \leq m} E_m(\mathbf{p}, h, \mathfrak{f}, n).$$



When we apply the last proposition to the function  $h(x) = 1$ , we obtain

$$(16) \quad S^L(n\mathbf{p}, 1) = \sum_{\mathbf{f} \in \mathcal{F}(\mathbf{p})} \mu^L(n\mathbf{t}(\mathbf{p}, \mathbf{f}))(0) \operatorname{vol}(\mathbf{f}) n^{\dim \mathbf{f}}.$$

As  $\mu^L(n\mathbf{t}(\mathbf{p}, \mathbf{f}))$  is invariant by integral translations, the function  $\mu^L(n\mathbf{t}(\mathbf{p}, \mathbf{f}))(0)$  is of period  $q_{\mathbf{f}}$ .

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